

$$\begin{aligned}
 Z &= \sqrt{R^2 + (X_L - X_C)^2} \\
 &= \left\{ (5.00 \, \Omega)^2 + \left[(1.00 \times 10^4 \, \text{rad/s})(2.00 \times 10^{-3} \, \text{H}) - \frac{1}{(1.00 \times 10^4 \, \text{rad/s})(4.00 \times 10^{-6} \, \text{F})} \right]^2 \right\}^{1/2} \\
 &= 7.07 \, \Omega.
 \end{aligned}$$

c. From **Equation 15.14**, the average power transferred to the circuit is

$$P_{\text{ave}} = \frac{V_{\text{rms}}^2 R}{Z^2} = \frac{(2.83 \, \text{V})^2 (5.00 \, \Omega)}{(7.07 \, \Omega)^2} = 0.801 \, \text{W}.$$

Significance

If the resistance is much larger than the reactance of the capacitor or inductor, the average power is a dc circuit equation of $P = V^2/R$, where V replaces the rms voltage.



15.4 Check Your Understanding An ac voltmeter attached across the terminals of a 45-Hz ac generator reads 7.07 V. Write an expression for the emf of the generator.



15.5 Check Your Understanding Show that the rms voltages across a resistor, a capacitor, and an inductor in an ac circuit where the rms current is I_{rms} are given by $I_{\text{rms}}R$, $I_{\text{rms}}X_C$, and $I_{\text{rms}}X_L$, respectively. Determine these values for the components of the RLC circuit of **Equation 15.12**.

15.5 | Resonance in an AC Circuit

Learning Objectives

By the end of the section, you will be able to:

- Determine the peak ac resonant angular frequency for a RLC circuit
- Explain the width of the average power versus angular frequency curve and its significance using terms like bandwidth and quality factor

In the RLC series circuit of **Figure 15.11**, the current amplitude is, from **Equation 15.10**,

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}. \quad (15.15)$$

If we can vary the frequency of the ac generator while keeping the amplitude of its output voltage constant, then the current changes accordingly. A plot of I_0 versus ω is shown in **Figure 15.17**.

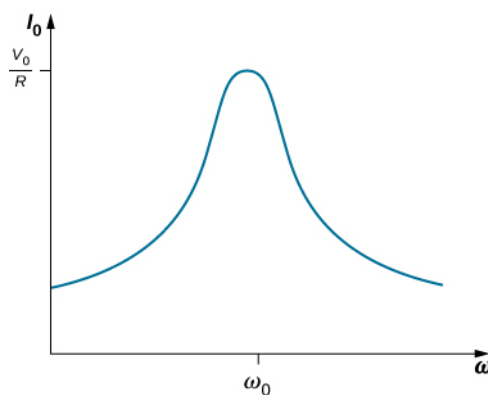


Figure 15.17 At an RLC circuit's resonant frequency, $\omega_0 = \sqrt{1/LC}$, the current amplitude is at its maximum value.

In **Oscillations** (<http://cnx.org/content/m58360/latest/>), we encountered a similar graph where the amplitude of a damped harmonic oscillator was plotted against the angular frequency of a sinusoidal driving force (see **Forced Oscillations** (http://cnx.org/content/m58366/latest/#CNX_UPhysics_15_07_ForcDmpAmp)). This similarity is more than just a coincidence, as shown earlier by the application of Kirchhoff's loop rule to the circuit of **Figure 15.11**. This yields

$$L \frac{di}{dt} + iR + \frac{q}{C} = V_0 \sin \omega t, \quad (15.16)$$

or

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = V_0 \sin \omega t,$$

where we substituted $dq(t)/dt$ for $i(t)$. A comparison of **Equation 15.16** and, from **Oscillations** (<http://cnx.org/content/m58360/latest/>), **Damped Oscillations** (<http://cnx.org/content/m58365/latest/#fs-id1167131231570>) for damped harmonic motion clearly demonstrates that the driven RLC series circuit is the electrical analog of the driven damped harmonic oscillator.

The **resonant frequency** f_0 of the RLC circuit is the frequency at which the amplitude of the current is a maximum and the circuit would oscillate if not driven by a voltage source. By inspection, this corresponds to the angular frequency $\omega_0 = 2\pi f_0$ at which the impedance Z in **Equation 15.15** is a minimum, or when

$$\omega_0 L = \frac{1}{\omega_0 C}$$

and

$$\omega_0 = \sqrt{\frac{1}{LC}}. \quad (15.17)$$

This is the resonant angular frequency of the circuit. Substituting ω_0 into **Equation 15.9**, **Equation 15.10**, and **Equation 15.11**, we find that at resonance,

$$\phi = \tan^{-1}(0) = 0, \quad I_0 = V_0/R, \quad \text{and} \quad Z = R.$$

Therefore, at resonance, an RLC circuit is purely resistive, with the applied emf and current in phase.

What happens to the power at resonance? **Equation 15.14** tells us how the average power transferred from an ac generator to the RLC combination varies with frequency. In addition, P_{ave} reaches a maximum when Z , which depends on the frequency, is a minimum, that is, when $X_L = X_C$ and $Z = R$. Thus, at resonance, the average power output of the source

in an RLC series circuit is a maximum. From **Equation 15.14**, this maximum is V_{rms}^2/R .

Figure 15.18 is a typical plot of P_{ave} versus ω in the region of maximum power output. The **bandwidth** $\Delta\omega$ of the resonance peak is defined as the range of angular frequencies ω over which the average power P_{ave} is greater than one-half the maximum value of P_{ave} . The sharpness of the peak is described by a dimensionless quantity known as the **quality factor** Q of the circuit. By definition,

$$Q = \frac{\omega_0}{\Delta\omega}, \quad (15.18)$$

where ω_0 is the resonant angular frequency. A high Q indicates a sharp resonance peak. We can give Q in terms of the circuit parameters as

$$Q = \frac{\omega_0 L}{R}. \quad (15.19)$$

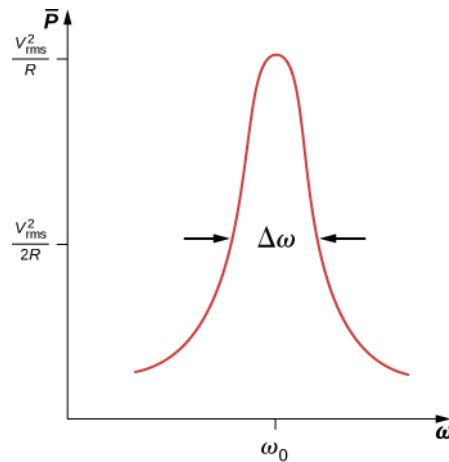


Figure 15.18 Like the current, the average power transferred from an ac generator to an RLC circuit peaks at the resonant frequency.

Resonant circuits are commonly used to pass or reject selected frequency ranges. This is done by adjusting the value of one of the elements and hence “tuning” the circuit to a particular resonant frequency. For example, in radios, the receiver is tuned to the desired station by adjusting the resonant frequency of its circuitry to match the frequency of the station. If the tuning circuit has a high Q , it will have a small bandwidth, so signals from other stations at frequencies even slightly different from the resonant frequency encounter a high impedance and are not passed by the circuit. Cell phones work in a similar fashion, communicating with signals of around 1 GHz that are tuned by an inductor-capacitor circuit. One of the most common applications of capacitors is their use in ac-timing circuits, based on attaining a resonant frequency. A metal detector also uses a shift in resonance frequency in detecting metals (**Figure 15.19**).



Figure 15.19 When a metal detector comes near a piece of metal, the self-inductance of one of its coils changes. This causes a shift in the resonant frequency of a circuit containing the coil. That shift is detected by the circuitry and transmitted to the diver by means of the headphones. (credit: modification of work by Eric Lippmann, U.S. Navy)

Example 15.4

Resonance in an RLC Series Circuit

(a) What is the resonant frequency of the circuit of **Example 15.1**? (b) If the ac generator is set to this frequency without changing the amplitude of the output voltage, what is the amplitude of the current?

Strategy

The resonant frequency for a RLC circuit is calculated from **Equation 15.17**, which comes from a balance between the reactances of the capacitor and the inductor. Since the circuit is at resonance, the impedance is equal to the resistor. Then, the peak current is calculated by the voltage divided by the resistance.

Solution

- a. The resonant frequency is found from **Equation 15.17**:

$$\begin{aligned} f_0 &= \frac{1}{2\pi} \sqrt{\frac{1}{LC}} = \frac{1}{2\pi} \sqrt{\frac{1}{(3.00 \times 10^{-3} \text{ H})(8.00 \times 10^{-4} \text{ F})}} \\ &= 1.03 \times 10^2 \text{ Hz.} \end{aligned}$$

- b. At resonance, the impedance of the circuit is purely resistive, and the current amplitude is

$$I_0 = \frac{0.100 \text{ V}}{4.00 \Omega} = 2.50 \times 10^{-2} \text{ A.}$$

Significance

If the circuit were not set to the resonant frequency, we would need the impedance of the entire circuit to calculate the current.

Example 15.5

Power Transfer in an RLC Series Circuit at Resonance

(a) What is the resonant angular frequency of an RLC circuit with $R = 0.200\ \Omega$, $L = 4.00 \times 10^{-3}\ \text{H}$, and $C = 2.00 \times 10^{-6}\ \text{F}$? (b) If an ac source of constant amplitude $4.00\ \text{V}$ is set to this frequency, what is the average power transferred to the circuit? (c) Determine Q and the bandwidth of this circuit.

Strategy

The resonant angular frequency is calculated from **Equation 15.17**. The average power is calculated from the rms voltage and the resistance in the circuit. The quality factor is calculated from **Equation 15.19** and by knowing the resonant frequency. The bandwidth is calculated from **Equation 15.18** and by knowing the quality factor.

Solution

a. The resonant angular frequency is

$$\begin{aligned}\omega_0 &= \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(4.00 \times 10^{-3}\ \text{H})(2.00 \times 10^{-6}\ \text{F})}} \\ &= 1.12 \times 10^4\ \text{rad/s}.\end{aligned}$$

b. At this frequency, the average power transferred to the circuit is a maximum. It is

$$P_{\text{ave}} = \frac{V_{\text{rms}}^2}{R} = \frac{[(1/\sqrt{2})(4.00\ \text{V})]^2}{0.200\ \Omega} = 40.0\ \text{W}.$$

c. The quality factor of the circuit is

$$Q = \frac{\omega_0 L}{R} = \frac{(1.12 \times 10^4\ \text{rad/s})(4.00 \times 10^{-3}\ \text{H})}{0.200\ \Omega} = 224.$$

We then find for the bandwidth

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{1.12 \times 10^4\ \text{rad/s}}{224} = 50.0\ \text{rad/s}.$$

Significance

If a narrower bandwidth is desired, a lower resistance or higher inductance would help. However, a lower resistance increases the power transferred to the circuit, which may not be desirable, depending on the maximum power that could possibly be transferred.



15.6 Check Your Understanding In the circuit of **Figure 15.11**, $L = 2.0 \times 10^{-3}\ \text{H}$, $C = 5.0 \times 10^{-4}\ \text{F}$, and $R = 40\ \Omega$. (a) What is the resonant frequency? (b) What is the impedance of the circuit at resonance? (c) If the voltage amplitude is $10\ \text{V}$, what is $i(t)$ at resonance? (d) The frequency of the AC generator is now changed to $200\ \text{Hz}$. Calculate the phase difference between the current and the emf of the generator.



15.7 Check Your Understanding What happens to the resonant frequency of an RLC series circuit when the following quantities are increased by a factor of 4: (a) the capacitance, (b) the self-inductance, and (c) the resistance?



15.8 Check Your Understanding The resonant angular frequency of an RLC series circuit is $4.0 \times 10^2\ \text{rad/s}$. An ac source operating at this frequency transfers an average power of $2.0 \times 10^{-2}\ \text{W}$ to the circuit. The resistance of the circuit is $0.50\ \Omega$. Write an expression for the emf of the source.